

New characterization of two-state normal distribution *

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Abstract. In this article we give a purely noncommutative criterion for the characterization of two-state normal distribution. We prove that families of two-state normal distribution can be described by relations which is similar to the conditional expectation in free probability, but has no classical analogue. We also show a generalization of Bożejko, Leinert and Speichers formula from [10] (relating moments and noncommutative cumulants).

Key words: generalized two-state freeness; generalized free Meixner distribution, conditional expectation; Laha-Lukacs theorem, noncommutative quadratic regression.

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1 Introduction

The original motivation for this paper is a desire to understand the results about two state conditional expectation which were shown in [13]. They proved that if the “outer” state satisfies the condition quadratic conditional variances, then moment generating function satisfies some relation. An open problem in this area is a converse implication of their theorem. We will show that this relation satisfies the condition from Theorem 2.1 of [13] and we extend this theorem (but that is not the main purpose of this article). The main goal of this paper is to construct a new condition connected with the “outer” state which gives us a new characterization of two-state normal distribution.

The study of a random variable in conditionally free probability has been an active research field during the last decade - see works [4, 5, 6, 7, 11, 13, 19, 23, 24, 25, 31, 32, 34, 38]. It is common in conditionally free probability, that its properties, to a large extent, are analogous to those of the classical and free probability. The main aim of this paper is to produce a new characterization of the two state normal laws, which is close to the quadratic regression property, but with no analog to the classical condition. As an example, consider two random variables which have the same distribution (because the result is more transparent with this assumption). Suppose that \mathbb{X} , \mathbb{Y} are c -free, self-adjoint, non-degenerate, centered and have the same distribution. Then \mathbb{X} and \mathbb{Y} have two-state normal laws (with respect to some states (φ, ψ) , which we will discuss later in section 2) if and only if there exist constants a, b such that

$$\varphi((\mathbb{X} - \mathbb{Y})^2 \mathbb{S}^n) = b\varphi(\mathbb{S}^n), \quad (1.1)$$

$$\varphi((\mathbb{X} - \mathbb{Y})\mathbb{S}^n(\mathbb{X} - \mathbb{Y})) = \varphi([(1 - b)\mathbb{S}^2 + (-2a + ab)\mathbb{S} + (a^2 + b^2)\mathbb{I}]\mathbb{S}^n), \quad (1.2)$$

where $\mathbb{S} = \mathbb{X} + \mathbb{Y}$. This result is unexpected because in commutative and free probability we have

$$\varphi((\mathbb{X} - \mathbb{Y})^2 \mathbb{S}^n) = \varphi((\mathbb{X} - \mathbb{Y})\mathbb{S}^n(\mathbb{X} - \mathbb{Y})),$$

for any classical and free variables \mathbb{X} and \mathbb{Y} . We also show that equation (1.1) is equivalent with two different conditions.

At this point it is worth mentioning about the characterization of type Laha-Lukacs in noncommutative and classical probability. In [26, 37], all the classical random variables and processes of Meixner type using a quadratic regression property were characterized. In free probability Bożejko, Bryc and Ejsmont proved that the first conditional linear moment and conditional quadratic variances characterize free Meixner laws (Bożejko and Bryc [12], Ejsmont [17]). Laha-Lukacs type characterizations of random variables in free probability are also studied by Szpojankowski, Wesołowski [36]. They give a characterization of noncommutative free-Poisson and free-Binomial variables by properties of the first two conditional moments, which mimics Lukacs type assumptions known from classical probability. Similar results have been obtained in boolean probability by Anshelevich [3]. He showed that in the boolean theory the Laha-Lukacs property characterizes only the Bernoulli distributions. It is worthwhile to mention the work of Bryc [14], where the Laha-Lukacs property for q -Gaussian processes was shown. Bryc proved that classical processes corresponding to operators which satisfy a q -commutation relations, have linear regressions and quadratic conditional variances.

The paper is organized as follows. In section 2 we review basic conditionally free probability, two-state normal laws and the statement of the main result. Next in the third section we quote complementary facts, lemmas and indications. In the fourth section we look more closely at non-crossing partitions with the first and last elements in the same block. In this section we also give extended version of Theorem 2.1 of [13] (Theorem 4.6) and generalize the Bożejko, Leinert and Speicher’s identity. Finally, in section 5 we prove our main results.

2 Basic facts about two-state freeness condition

Let \mathcal{A} be a unital $*$ -algebra with two-states $\varphi, \psi : \mathcal{A} \rightarrow \mathbb{C}$. We assume that states φ fulfill the usual assumptions of positivity and normalization, and we assume the tracial property $\psi(ab) = \psi(ba)$ for ψ , but not for φ . A typical model of an algebra with two-states is a group algebra of a group $G = *G_i$, where $*$ is a free product of groups G_i . Here φ is the boolean product of the individual states, the simplest example is the free product of integers, $G_i = \mathbb{X}$, where G_i is a free group with an arbitrary number of generators, and φ is the Haagerup state, $\phi(x) = r^{|x|}$, where $|x|$ is the length of word $x \in G$, $-1 \leq r \leq 1$, and state ψ is δ_e . For the details see [8, 9, 13].

A self-adjoint element $\mathbb{X} \in \mathcal{A}$ with moments that fulfill appropriate growth condition defines a pair (μ, ν) of probability measures on \mathbb{R} such that

$$\varphi(\mathbb{X}^n) = \int_{\mathbb{R}} x^n \mu(dx) \text{ and } \psi(\mathbb{X}^n) = \int_{\mathbb{R}} x^n \nu(dx). \quad (2.1)$$

We will refer to the measures μ, ν as the φ -law and the ψ -law of \mathbb{X} , respectively. In this paper we assume that μ and ν are compactly supported probability measures, so moments do not grow faster than exponentially.

Definition 2.1. Let $\pi = \{V_1, \dots, V_p\}$ be a partition of the linear ordered set $1, \dots, n$, i.e. the $V_i \neq \emptyset$ are ordered and disjoint sets whose union is $\{1, \dots, n\}$. Then π is called non-crossing if $a, c \in V_i$ and $b, d \in V_j$ with $a < b < c < d$ implies $i = j$.

The sets $V_i \in \pi$ are called blocks. In a non-crossing partition π , a block V_i is inner if for some $a, b \notin V_i$ (where a and b are in some other block of the partition π) and all $x \in V_i$, $a < x < b$, otherwise it is called outer. Family of all outer (resp. inner) blocks of π will be denoted by $Out(\pi)$ (resp. $Inn(\pi)$). We will denote the set of all non-crossing partitions of the set $\{1, \dots, n\}$ by $NC(n)$.

Definition 2.2. The free (non-crossing) cumulants are the k -linear maps $r_k : \mathcal{A}^k \rightarrow \mathbb{C}$ ($r_k = r_k^\psi = r_k^\nu$) defined by the recursive formula (connecting them with mixed moments see [33])

$$\psi(\mathbb{X}_1 \mathbb{X}_2 \dots \mathbb{X}_n) = \sum_{\nu \in NC(n)} r_\nu(\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n), \quad (2.2)$$

where

$$r_\nu(\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n) := \prod_{B \in \nu} r_{|B|}(\mathbb{X}_i : i \in B), \quad (2.3)$$

where $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n \in \mathcal{A}$. With each set of $\mathbb{X}_1, \dots, \mathbb{X}_n \in \mathcal{A}$ and a pair of states (φ, ψ) we associate the two-state free cumulants $R_k = R_k^{(\varphi, \psi)} = R_k^{(\mu, \nu)}$, $k = 1, 2, \dots$, which are multilinear functions $R_k : \mathcal{A}^k \rightarrow \mathbb{C}$ defined by

$$\varphi(\mathbb{X}_1 \dots \mathbb{X}_n) = \sum_{k=1}^n \sum_{s_1=1 < s_2 < \dots < s_k \leq n} R_k(\mathbb{X}_{s_1}, \dots, \mathbb{X}_{s_k}) \varphi(\mathbb{X}_{s_k+1} \dots \mathbb{X}_{s_n}) \prod_{r=1}^{k-1} \psi\left(\prod_{j=s_r+1}^{s_{r+1}-1} \mathbb{X}_j\right). \quad (2.4)$$

The above equation is equivalent to

$$\varphi(\mathbb{X}_1 \dots \mathbb{X}_n) = \sum_{\nu \in NC(n)} \prod_{B \in Out(\nu)} R_{|B|}(\mathbb{X}_i : i \in B) \prod_{B \in Inn(\nu)} r_{|B|}(\mathbb{X}_i : i \in B). \quad (2.5)$$

Sometimes we will write $r_k(\mathbb{X}) = r_k(\mathbb{X}, \dots, \mathbb{X})$ and $R_k(\mathbb{X}) = R_k(\mathbb{X}, \dots, \mathbb{X})$. Fix $\mathbb{X} \in \mathcal{A}$ and consider the following power series

$$\begin{aligned} r(z) &= r_\nu(z) = \sum_{i=0}^{\infty} r_{i+1}(\mathbb{X}, \dots, \mathbb{X}) z^i, \\ R(z) &= R_{\mathbb{X}}(z) = R_{(\mu, \nu)}(z) = \sum_{i=0}^{\infty} R_{i+1}(\mathbb{X}, \dots, \mathbb{X}) z^i, \\ M_\nu(z) &= \sum_{i=0}^{\infty} z^i \psi(\mathbb{X}^i), \\ M_\mu(z) &= \sum_{i=0}^{\infty} z^i \varphi(\mathbb{X}^i). \end{aligned}$$

For our purposes, the most convenient definition is the following. The Cauchy-Stieltjes transform of μ can be expanded into the following formal power series

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - y} \mu(dy) = \sum_{n=0}^{\infty} m_n(\mu) \frac{1}{z^{n+1}} = \frac{1}{z} M_\mu\left(\frac{1}{z}\right), \quad (2.6)$$

where $m_n(\mu)$ is the n -th moment of μ . Bellow we introduce a definition of free independence (see [21, 16, 22, 23, 24, 27, 28, 29, 30]).

Definition 2.3. (A) We say that subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots$ are ψ -free if for every choice of $i_1 \neq i_2 \dots \neq i_n$ and every choice of $\mathbb{X}_i \in \mathcal{A}_i$ such that $\psi(\mathbb{X}_i) = 0$ we have

$$\psi(\mathbb{X}_1 \mathbb{X}_2 \dots \mathbb{X}_n) = 0. \quad (2.7)$$

(B) This family is c -freely independent if it is ψ -freely independent and, under the same assumptions on $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_n$ also

$$\varphi(\mathbb{X}_1 \mathbb{X}_2 \dots \mathbb{X}_n) = \prod_{k=1}^n \varphi(\mathbb{X}_k). \quad (2.8)$$

The above definition is equivalent to the following.

Definition 2.4. We say that subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots$ are c -free if for every choice of $\mathbb{X}_1, \dots, \mathbb{X}_n \in \bigcup_j \mathcal{A}_j$ we have

$$r_n(\mathbb{X}_1, \dots, \mathbb{X}_n) = 0 \text{ and } R_n(\mathbb{X}_1, \dots, \mathbb{X}_n) = 0 \text{ except if all } \mathbb{X}_j \text{ come from the same algebra.}$$

Remark 2.5. It is important to note that Bożejko and Bryc use the following definition of independence:

We say that subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots$ are (φ, ψ) -free if for every choice of $\mathbb{X}_1, \dots, \mathbb{X}_n \in \bigcup_j \mathcal{A}_j$ we have

$$R_n(\mathbb{X}_1, \dots, \mathbb{X}_n) = 0 \text{ except if all } \mathbb{X}_j \text{ come from the same algebra.} \quad (2.9)$$

It is important to note that (φ, ψ) -freeness is weaker than c -freeness. The c -freeness implies (φ, ψ) -freeness – see Lemma 1.1 from [13].

Definition 2.6 (*c*-free convolution). *Using free cumulants, we can define in a uniform way the free convolution \boxplus for example:*

$$r_n^{\nu_1 \boxplus \nu_2} = r_n^{\nu_1} + r_n^{\nu_2}. \quad (2.10)$$

The two-state free (or conditionally free; c-free convolution; these terms will be used interchangeably) convolution \boxplus_c is an operation on pairs of measures, defined as follows: $(\mu_3, \nu_3) = (\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2)$ if and only if $\nu_3 = \nu_1 \boxplus \nu_2$ and

$$R_n^{(\mu_3, \nu_3)} = R_n^{(\mu_1, \nu_1)} + R_n^{(\mu_2, \nu_2)}. \quad (2.11)$$

2.1 Two-state normal distribution and the main result

Any probability measure μ on the real line, all of whose moments are finite, has two associated sequences of Jacobi parameters α_i, β_i for example, μ is the spectral measure of the tridiagonal matrix

$$\begin{pmatrix} \alpha_0 & \beta_0 & 0 & 0 & \ddots \\ 1 & \alpha_1 & \beta_1 & 0 & \ddots \\ 0 & 1 & \alpha_2 & \beta_2 & \ddots \\ 0 & 0 & 1 & \alpha_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (2.12)$$

We will denote this fact by

$$J(\mu) = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ \beta_0 & \beta_1 & \beta_2 & \dots \end{pmatrix} \quad (2.13)$$

with $\alpha_n(\mu) := \alpha_n, \beta_n(\mu) := \beta_n$. These parameters are related to the moments of the measure via the Accardi-Bożejko [1] formulas. If the measure μ has all moments, then by a theorem of Stieltjes (see [2]), it can be expressed as a continued fraction:

$$G_\mu(z) = \frac{1}{z - \alpha_0 - \frac{\beta_0}{z - \alpha_1 - \frac{\beta_1}{z - \alpha_2 - \frac{\beta_2}{\ddots}}}}. \quad (2.14)$$

If some $\beta_i = 0$ the continued fraction terminates, that is the subsequent α and β coefficients can be defined arbitrarily. See [15] for more details. The monic orthogonal polynomials P_n for μ satisfy the following recursion relation

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_{n-1} P_{n-1}(x), \quad (2.15)$$

with $P_{-1}(x) = 0$.

Definition 2.7. \mathbb{X} is a two-state normal (Gaussian) distribution if $R_k(\mathbb{X}) = 0$ and $r_k(\mathbb{X}) = 0$ for $k > 2$.

Without the loss of generality we can assume (in this paper), that two-state normal element \mathbb{X} has Jacobi parameters

$$J(\mu_{a,b}) = \begin{pmatrix} a, & 0, & 0, & 0, & \dots \\ b, & 1, & 1, & 1, & \dots \end{pmatrix}, \quad (2.16)$$

$$J(\nu) = \begin{pmatrix} 0, & 0, & 0, & 0, & \dots \\ 1, & 1, & 1, & 1, & \dots \end{pmatrix}, \quad (2.17)$$

which means that $\mu_{a,b}$ (where $a \in \mathbb{R}$ and $b > 0$ – this assumption on a and b will be valid till the end of the work) is free Meixner distribution and ν is normalized Wigner's semicircle law. The first cumulants of this distribution are as follows $R_1(\mathbb{X}) = a$, $R_2(\mathbb{X}) = b$, $r_1(\mathbb{X}) = 0$ and $r_2(\mathbb{X}) = 1$.

For particular values of a and b the law of $\mu_{a,b}$ is (see [12]):

- the Wigner's semicircle law if $a = 0$ and $b = 1$;
- the free Poisson law if $a \neq 0$ and $b = 1$;
- the free Pascal (negative binomial) type law if $b < 1$ and $a^2 > 4(1 - b)$;
- the free Gamma law if $b < 1$ and $a^2 = 4(1 - b)$;
- the pure free Meixner law if $b < 1$ and $a^2 < 4(1 - b)$;
- the free binomial law $b > 1$.

2.2 The main result

Now we can state the main result of this paper.

Theorem 2.8. *Suppose \mathbb{X}, \mathbb{Y} are c -free, self-adjoint, non-degenerate, $\psi(\mathbb{X}) = \psi(\mathbb{Y}) = 0$, $\psi(\mathbb{X}^2 + \mathbb{Y}^2) = 1$, $\varphi(\mathbb{X}) = \alpha a$, $\varphi(\mathbb{Y}) = \beta a$, $\varphi((\mathbb{X} + \mathbb{Y})^2) = b + a^2$, $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$, $\beta r_k(\mathbb{X}) = \alpha r_k(\mathbb{Y})$ for some $\alpha, \beta > 0$, $\alpha + \beta = 1$ and all integers $k \geq 1$. Let $\mathbb{S} = \mathbb{X} + \mathbb{Y}$, then the following statements are equivalent:*

1. \mathbb{X}, \mathbb{Y} have a two-state normal distribution,
- 2.

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2\mathbb{S}^n) = \alpha\beta b\varphi(\mathbb{S}^n), \quad (2.18)$$

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})\mathbb{S}^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta\varphi([(1-b)\mathbb{S}^2 + (-2a + ab)\mathbb{S} + (a^2 + b^2)\mathbb{I}]\mathbb{S}^n), \quad (2.19)$$

3. (2.19) and

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})\mathbb{S}^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta b\psi(\mathbb{S}^n), \quad (2.20)$$

4. (2.19) and

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})\mathbb{S}^n) = \alpha\beta a\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2\mathbb{S}^n), \quad (2.21)$$

where all above relations hold for all non-negative integers $n \geq 0$.

Remark 2.9. *In free probability we can formulate the following theorem (see [12, 17]).*

Theorem 2.10. Suppose that \mathbb{X}, \mathbb{Y} are free, self-adjoint, non-degenerate $\varphi(\mathbb{X}) = \alpha a, \varphi(\mathbb{Y}) = \beta a$ and $\varphi(\mathbb{X}^2 + \mathbb{Y}^2) = b + a^2$. Then $\mathbb{X}/\sqrt{\alpha}$ and $\mathbb{Y}/\sqrt{\beta}$ have the free Meixner laws $\mu_{a\sqrt{\alpha}, b}$ and $\mu_{a\sqrt{\beta}, b}$, respectively, where $\alpha + \beta = 1, a \in \mathbb{R}, b > 0$ if and only if

$$\varphi(\mathbb{X}|\mathbb{X} + \mathbb{Y}) = \alpha(\mathbb{X} + \mathbb{Y}) + a\mathbb{I}, \quad (2.22)$$

$$\text{Var}(\mathbb{X}|\mathbb{X} + \mathbb{Y}) = \alpha\beta((1-b)(\mathbb{X} + \mathbb{Y})^2 + (-2a + ab)(\mathbb{X} + \mathbb{Y}) + (a^2 + b^2)\mathbb{I}) + a\mathbb{I}. \quad (2.23)$$

We can easily show that equation (2.23) is equivalent to

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2|\mathbb{X} + \mathbb{Y}) = \alpha\beta((1-b)(\mathbb{X} + \mathbb{Y})^2 + (-2a + ab)(\mathbb{X} + \mathbb{Y}) + (a^2 + b^2)\mathbb{I}),$$

so we see that condition (2.19) behaves like conditional variances in free probability.

3 Complementary facts, lemmas and indications

Definition 3.1. We introduce the notation

$$\varphi_k(\mathbb{X}^n) = \sum_{j=k}^n \sum_{s_1=1 < s_2=2 < \dots < s_k=k < \dots < s_j \leq n} R_j(\mathbb{X}) \varphi(\mathbb{X}^{s_n-1-s_j}) \prod_{r=k}^{j-1} \psi(\mathbb{X}^{s_{r+1}-1-s_r}), \quad (3.1)$$

where $k \leq n$. The above equation corresponds to moments of φ with the first k elements in the same block. Analogously we can define $\varphi_k(\mathbb{X}_1 \dots \mathbb{X}_n)$.

Example 3.2. For $k = 3$ and $n = 5$, we get:

$$\varphi_3(\mathbb{X}^5) = R_3(\mathbb{X})\varphi(\mathbb{X}^2) + R_4(\mathbb{X})\varphi(\mathbb{X}) + R_4(\mathbb{X})\psi(\mathbb{X}) + R_5(\mathbb{X}). \quad (3.2)$$

The following lemma is a two-state version of Lemma 2.4 in [18] (the proof is also similar).

Lemma 3.3. Let \mathbb{X} be a self-adjoint element of the algebra \mathcal{A} then

$$\varphi_k(\mathbb{X}^{n+k}) = \sum_{j=1}^n \psi(\mathbb{X}^{j-1}) \varphi_{k+1}(\mathbb{X}^{n+k-j+1}) + R_k(\mathbb{X})\varphi(\mathbb{X}^{n-k}), \quad (3.3)$$

where $k, n \geq 1$ and we take convention $\psi(\mathbb{X}^0) = 1$.

Proof. First, we will consider partitions with the first k elements in the same block, i.e. we sum only for $j = k$ in the equation (3.1) which corresponds to $R_k(\mathbb{X})\varphi(\mathbb{X}^{n-k})$.

On the other hand, for $j > k$ denote $s(\nu) = \min\{j : j > k, j \in B_1\}$ where B_1 is the block which contains $1, \dots, k$ (in Figure 1 it is an element j). This decomposes our situation into the n classes which can be identified with the product $\psi(\mathbb{X}^{j-k-1}) \times \varphi_{k+1}(\mathbb{X}^{n+2k-j+1})$ where $j \in \{k+1, \dots, n+k\}$. Indeed, the blocks in which partitions are the elements $\{k+1, \dots, j-1\}$ can be identified with $\psi(\mathbb{X}^{j-k-1})$ (in Figure 1 these are stars), and under the additional constraint that the first $k+1$ elements are in the same block, the remaining blocks, which are partitions of the set $\{1, \dots, k, j, j+1, \dots, n+k\}$, can be uniquely identified with $\varphi_{k+1}(\mathbb{X}^{n+2k-j+1})$ (in Figure 1 it is a part without the stars). This gives the formula (3.3) (if we re-index j) and proves the lemma.

□



Figure 1: The main structure of non-crossing partitions of $\{1, 2, 3, \dots, n+k\}$ with the first k elements in the same block.

Definition 3.4. Let \mathbb{X} be a (self-adjoint) element of the algebra \mathcal{A} . We introduce functions (series):

$$C_{\varphi, \psi}^{(k)}(z) = \sum_{n=0}^{\infty} \varphi_k(\mathbb{X}^{k+n}) z^{k+n}, \text{ where } k \geq 1 \quad (3.4)$$

for sufficiently small $|z|$ and $z \in \mathbb{C}$. This series is convergent because we consider such a series as $M_{\nu_{\mathbb{X}}}(z)_{\mathbb{X}}$ and $M_{\mu_{\mathbb{X}}}(z)$ is convergent for sufficiently small $|z|$. Thus from Lemma 3.3 we get that $C_{\varphi, \psi}^{(2)}(z)$ is convergent because $C_{\varphi, \psi}^{(1)}(z) = M_{\mu_{\mathbb{X}}}(z) - 1$. For $k > 2$ this is immediate, by induction on k and by using the preceding lemma.

Lemma 3.5. Let \mathbb{X} be a (self-adjoint) element of the algebra \mathcal{A} , then

$$C_{\varphi, \psi}^{(k)}(z) = M_{\nu}(z) C_{\varphi, \psi}^{(k+1)}(z) + R_k(\mathbb{X}) z^k M_{\mu}(z). \quad (3.5)$$

Proof. It is clear from Lemma 3.3 that we have

$$\begin{aligned} C_{\varphi, \psi}^{(k)}(z) &= \sum_{n=0}^{\infty} \varphi_k(\mathbb{X}^{k+n}) z^{k+n} = \varphi_k(\mathbb{X}^k) z^k + \sum_{n=1}^{\infty} \varphi_k(\mathbb{X}^{n+k}) z^{k+n} \\ &= \varphi_k(\mathbb{X}^k) z^k + \sum_{n=1}^{\infty} \left[\sum_{i=0}^{n-1} \psi(\mathbb{X}^i) \varphi_{k+1}(\mathbb{X}^{n+k-i}) + R_k(\mathbb{X}) \varphi(\mathbb{X}^n) \right] z^{k+n} \\ &= \varphi_k(\mathbb{X}^k) z^k + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \psi(\mathbb{X}^i) z^i \varphi_{k+1}(\mathbb{X}^{n+k-i}) z^{k+n-i} + R_k(\mathbb{X}) z^k \sum_{n=1}^{\infty} \varphi(\mathbb{X}^n) z^n \\ &= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \psi(\mathbb{X}^i) z^i \varphi_{k+1}(\mathbb{X}^{n+k-i}) z^{k+n-i} + R_k(\mathbb{X}) z^k \sum_{n=0}^{\infty} \varphi(\mathbb{X}^n) z^n \\ &= M_{\nu}(z) C_{\varphi, \psi}^{(k+1)}(z) + R_k(\mathbb{X}) z^k M_{\mu}(z), \end{aligned} \quad (3.6)$$

which proves the lemma. \square

Example 3.6. For $k = 1$, we get

$$C_{\varphi, \psi}^{(1)}(z) = M_{\mu}(z) - 1 = M_{\nu}(z) C_{\varphi, \psi}^{(2)}(z) + R_1(\mathbb{X}) z M_{\mu}(z). \quad (3.7)$$

Similarly, by putting $k = 2$, we obtain

$$C_{\varphi, \psi}^{(2)}(z) = M_{\nu}(z) C_{\varphi, \psi}^{(3)}(z) + R_2(\mathbb{X}) z^2 M_{\mu}(z). \quad (3.8)$$

Definition 3.7. We introduce the notation

$$\varphi_{\parallel}(\mathbb{X}^n) = \sum_{k=1}^n \sum_{s_1=1 < s_2 < \dots < s_k=n} R_k(\mathbb{X}^k) \prod_{r=1}^{k-1} \psi(\mathbb{X}^{s_{r+1}-1-s_r}), \quad (3.9)$$

where $n \geq 2$. The above equation corresponds to “moments” of φ with the first and last element in the same block.

Example 3.8. For $n = 5$, we get

$$\varphi_{\parallel}(\mathbb{X}^5) = R_2(\mathbb{X})\psi(\mathbb{X}^3) + 2R_3(\mathbb{X})\psi(\mathbb{X}^2) + R_3(\mathbb{X})\psi^2(\mathbb{X}) + 3R_4(\mathbb{X})\psi(\mathbb{X}) + R_5(\mathbb{X}). \quad (3.10)$$

Lemma 3.9. Let \mathbb{X} be the self-adjoint element of the algebra \mathcal{A} , then

$$\varphi(\mathbb{X}^n) = \sum_{j=2}^n \varphi_{\parallel}(\mathbb{X}^j) \varphi(\mathbb{X}^{n-j}) + R_1(\mathbb{X}) \varphi(\mathbb{X}^{n-1}), \quad (3.11)$$

where $n \geq 1$.

Proof. The proof is based on the analysis of the formula (2.4). First, we will consider partitions with singleton 1, i.e. $\pi = \{V_1, \dots, V_k\}$ where $V_1 = \{1\}$. It is clear that the sum over all non-crossing partitions of this form corresponds to the term $R_1(\mathbb{X})\varphi(\mathbb{X}^{n-1})$.

On the other hand, for such partitions as in notation (2.4) let $s = s(\nu) \in \{2, 3, \dots, n\}$ denote the most-right element of the block containing 1. This decomposes our situation into the $n - 1$ classes $s(\nu) = j$, $j \in \{2, \dots, n\}$. This set can be identified with the product $\varphi_{\parallel}(\mathbb{X}^j) \varphi(\mathbb{X}^{n-j})$. Indeed, the blocks which partition the elements $\{j + 1, \dots, n\}$ can be identified with $\varphi(\mathbb{X}^{n-j})$ (on Figure 2 it is the part on the right side of the element j), and under the additional constraint that is the first one (i.e. 1) and last element (i.e. j) are in the same block, the remaining blocks, which partition the set $\{1, \dots, j\}$, can be uniquely identified with $\varphi_{\parallel}(\mathbb{X}^j)$, it follows from the definition of φ_{\parallel} , i.e. equation (3.9) (in Figure 2 it is the block which contains 1 and j). This yields the formula (3.11) and proves the lemma.

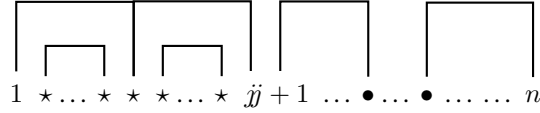


Figure 2: The main structure of the non-crossing partitions of $\{1, 2, 3, \dots, j, \dots, n\}$ with the first one and the j -th element in the same block.

□

Definition 3.10. Let \mathbb{X} be a (self-adjoint) element of the algebra \mathcal{A} . We introduce the following functions (series):

$$C_{\parallel}(z) = C_{\parallel, \varphi, \psi}(z) = \sum_{n=0}^{\infty} \varphi_{\parallel}(\mathbb{X}^{2+n}) z^{2+n}, \quad (3.12)$$

for sufficiently small z and $z \in \mathbb{C}$. This series is convergent by a similar argument as for $C_{\varphi, \psi}^{(k)}(z)$ (using Lemma 3.9).

Lemma 3.11. *Let \mathbb{X} be a (self-adjoint) element of the algebra \mathcal{A} , then*

$$M_\mu(z) - 1 = M_\mu(z)C_{\parallel}(z) + R_1(\mathbb{X})zM_\mu(z). \quad (3.13)$$

Proof. The proof is analogous to the proof of Lemma 3.5 and based on the equation (3.11). \square

Corollary 3.12. *We have the following identity*

$$M_\nu(z)C_{\varphi,\psi}^{(2)}(z) = M_\mu(z)C_{\parallel}(z). \quad (3.14)$$

Proof. Compare the equation (3.7) with the equation (3.13). \square

Lemma 3.13. *Suppose that \mathbb{X}, \mathbb{Y} are self-adjoint, c -free, $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ for some $\alpha, \beta > 0$ and all integer $k \geq 1$. Then*

$$R_k(\beta\mathbb{X} - \alpha\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}, \dots, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}) = 0, \quad (3.15)$$

$$R_k(\beta\mathbb{X} - \alpha\mathbb{Y}, \beta\mathbb{X} - \alpha\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \dots, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}) = \alpha\beta R_k(\mathbb{X} + \mathbb{Y}), \quad (3.16)$$

$$R_k(\beta\mathbb{X} - \alpha\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}, \dots, \mathbb{X} + \mathbb{Y}, \beta\mathbb{X} - \alpha\mathbb{Y}) = \alpha\beta R_k(\mathbb{X} + \mathbb{Y}). \quad (3.17)$$

Proof. By taking into account the fact that R_k are multilinear functions and from the assumption of c -free and $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ we get

$$R_k(\beta\mathbb{X} - \alpha\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \mathbb{X} + \mathbb{Y}, \dots, \mathbb{X} + \mathbb{Y}) = \beta R_k(\mathbb{X}) - \alpha R_k(\mathbb{Y}) = 0, \quad (3.18)$$

and similarly for $k \geq 2$

$$\begin{aligned} & R_k(\beta\mathbb{X} - \alpha\mathbb{Y}, \beta\mathbb{X} - \alpha\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \dots, \mathbb{X} + \mathbb{Y}) \\ &= \beta^2 R_k(\mathbb{X}) + \alpha^2 R_k(\mathbb{Y}) = \beta\alpha R_k(\mathbb{X} + \mathbb{Y}), \end{aligned} \quad (3.19)$$

$$\begin{aligned} & R_k(\beta\mathbb{X} - \alpha\mathbb{Y}, \mathbb{X} + \mathbb{Y}, \dots, \mathbb{X} + \mathbb{Y}, \beta\mathbb{X} - \alpha\mathbb{Y}) \\ &= \beta^2 R_k(\mathbb{X}) + \alpha^2 R_k(\mathbb{Y}) = \beta\alpha R_k(\mathbb{X} + \mathbb{Y}). \end{aligned} \quad (3.20)$$

and the assertion follows. \square

Lemma 3.14. *Suppose that \mathbb{X}, \mathbb{Y} are self-adjoint, c -free, $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ for some $\alpha, \beta > 0$ and all integer $k \geq 1$. Then*

1. $\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2(\mathbb{X} + \mathbb{Y})^n) = \alpha\beta\varphi_2((\mathbb{X} + \mathbb{Y})^{n+2}),$
2. $\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta\varphi_{\parallel}((\mathbb{X} + \mathbb{Y})^{n+2}).$

Proof. 1. Now we use the moment-cumulant formula (2.2) and (3.16)

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2(\mathbb{X} + \mathbb{Y})^n) = \varphi_2((\beta\mathbb{X} - \alpha\mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n) = \alpha\beta\varphi_2((\mathbb{X} + \mathbb{Y})^{n+2}), \quad (3.21)$$

because if the first element $\beta\mathbb{X} - \alpha\mathbb{Y}$ is in the partition with an element only from the “part” $(\mathbb{X} + \mathbb{Y})^n$ then we have (3.15) (the sum over this partition vanishes). Thus the first element and second one must be in the same block and taking into account the equation (3.16), we get (3.21).

2. Now we show

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta\varphi_n((\mathbb{X} + \mathbb{Y})^{n+2}). \quad (3.22)$$

We have that either the first and the last elements are in different blocks, or they are in the same block. In the first case, $\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = 0$ by the equation (3.15). On the other hand, if they are in the same block, then from the Lemma 3.13 we get (3.22). \square

Now we present a theorem which follows from the main result of [7]. It will be used in the proof of the main theorem in order to calculate the moment generating function of free convolution.

Theorem 3.15. *Let (μ, ν) be a pair of measures with Jacobi parameters (2.16) and (2.17), respectively. Then the conditionally free power $(\mu_t, \nu_t) = (\mu, \nu)^{\boxplus_{c^t}}$ exists for $t \geq 0$ and we have*

$$J(\mu_t) = \begin{pmatrix} at, & 0, & 0, & 0, & \dots \\ bt, & t, & t, & t, & \dots \end{pmatrix} \quad (3.23)$$

and

$$J(\nu_t) = \begin{pmatrix} 0, & 0, & 0, & 0, & \dots \\ t, & t, & t, & t, & \dots \end{pmatrix}. \quad (3.24)$$

Lemma 3.16. *Suppose \mathbb{X}, \mathbb{Y} are c -free and self-adjoint. Denote the distribution of $\mathbb{X}/\sqrt{\alpha}$ and $\mathbb{Y}/\sqrt{\beta}$ by (μ_1, ν_1) and (μ_2, ν_2) , respectively. We assume that Jacobi parameters for this measure are equal to*

$$J(\mu_1) = \begin{pmatrix} a\sqrt{\alpha}, & 0, & 0, & 0, & \dots \\ b, & 1, & 1, & 1, & \dots \end{pmatrix}, J(\mu_2) = \begin{pmatrix} a\sqrt{\beta}, & 0, & 0, & 0, & \dots \\ b, & 1, & 1, & 1, & \dots \end{pmatrix} \quad (3.25)$$

and

$$J(\nu_1) = \begin{pmatrix} 0, & 0, & 0, & 0, & \dots \\ 1, & 1, & 1, & 1, & \dots \end{pmatrix}, J(\nu_2) = \begin{pmatrix} 0, & 0, & 0, & 0, & \dots \\ 1, & 1, & 1, & 1, & \dots \end{pmatrix}. \quad (3.26)$$

where $\alpha, \beta > 0, \alpha + \beta = 1$. Then $\mathbb{X} + \mathbb{Y}$ has two-state normal law (μ, ν) with Jacobi parameters (2.16) and (2.17), respectively.

Proof. We use the following well-known fact if a certain variable \mathbb{X} has distribution with Jacobi parameters (2.13) then $\gamma\mathbb{X}$ (where $\gamma \in \mathbb{R}$) has the following Jacobi parameters (see [20])

$$\begin{pmatrix} \gamma\alpha_0, & \gamma\alpha_1, & \gamma\alpha_2, & \dots \\ \gamma^2\beta_0, & \gamma^2\beta_1, & \gamma^2\beta_2, & \dots \end{pmatrix}. \quad (3.27)$$

Thus we deduce that \mathbb{X} and \mathbb{Y} has respectively the following Jacobi parameters (with respect to the state φ)

$$\begin{pmatrix} a\alpha, & 0, & 0, & 0, & \dots \\ b\alpha, & \alpha, & \alpha, & \alpha, & \dots \end{pmatrix}, \begin{pmatrix} a\beta, & 0, & 0, & 0, & \dots \\ b\beta, & \beta, & \beta, & \beta, & \dots \end{pmatrix}.$$

Using Theorem 3.15 we deduce that the law of $\mathbb{X} + \mathbb{Y}$ is given by (2.16) with respect to the state φ . Analogously, we have that $\mathbb{X} + \mathbb{Y}$ has Jacobi parameters (2.17) with respect to the state ψ . \square

4 A new relation in conditionally free probability

4.1 A generalization of Bożejko, Leinert and Speicher's identity

From [10], we have the following relation

$$M_\mu(z)(1 - z\mathcal{R}_\mathbb{X}(zM_\nu(z))) = 1. \quad (4.1)$$

The relation (4.1) can be generalized as follows:

Proposition 4.1. *Suppose that \mathbb{X} is a self-adjoint element of the algebra \mathcal{A} , then*

$$C_{\varphi,\psi}^{(k)}(z) = \mathcal{R}_\mathbb{X}^{(k)}(zM_\nu(z))z^k M_\mu(z), \quad (4.2)$$

where $\mathcal{R}_\mathbb{X}^{(k)}(z) = \sum_{i=k}^{\infty} R_i(\mathbb{X})z^{i-k}$.

Proof. We prove this by the induction on k . The case $k = 1$ is clear because $C_\mu^{(1)}(z) = M_\mu(z) - 1$. The induction step $k \Rightarrow k + 1$ (for $k > 1$) follows immediately using Lemma 3.5 which gives

$$C_{\varphi,\psi}^{(k+1)}(z) = \frac{C_{\varphi,\psi}^{(k)}(z)}{M_\mu(z)} - R_k(\mathbb{X})z^k = \mathcal{R}_\mathbb{X}^{(k)}(zM_\nu(z))z^k - R_k(\mathbb{X})z^k \quad (4.3)$$

$$= \mathcal{R}_\mathbb{X}^{(k+1)}(zM_\nu(z))z^{k+1} M_\mu(z). \quad (4.4)$$

\square

4.2 A new relation between moments

In this subsection we explain the motivation for introducing φ_{II} from the point of view of two-state free probability. The answer to this problem turns out to be the following: the relation between measures whose Jacobi parameters are described by (2.13) and other measure whose Jacobi parameter equals

$$\begin{pmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \dots \\ \beta_1, & \beta_2, & \beta_3, & \dots \end{pmatrix}, \quad (4.5)$$

is contained in φ_{II} .

Theorem 4.2. *Suppose that \mathbb{X} and \mathbb{Y} are self-adjoint elements of algebra \mathcal{A} . Denote by μ the distribution of \mathbb{X} with respect to φ , and by ρ the distribution of \mathbb{Y} with respect to φ (the distribution of the state ψ is irrelevant in this theorem). If measure μ has Jacobi parameters described by (2.13) where $\beta_0 > 0$, then the relation between the measure ρ of the variable \mathbb{Y} described by the parameter (4.5) is given by*

$$\varphi_{\text{II}}(\mathbb{X}^{n+2}) = \beta_0 \varphi(\mathbb{Y}^n), \quad (4.6)$$

for all $n \geq 0$.

Proof. From (2.14) we have

$$G_\mu(z) = \frac{1}{z - \alpha_0 - \beta_0 G_\rho(z)}. \quad (4.7)$$

By using the relations $M_\mu(z) = \frac{1}{z} G_\mu(\frac{1}{z})$ and $M_\rho(z) = \frac{1}{z} G_\rho(\frac{1}{z})$ we see that

$$M_\mu(z)(1 - z\alpha_0 - \beta_0 z^2 M_\rho(z)) = 1. \quad (4.8)$$

Applying Lemma 3.11 we get

$$M_\mu(z) - 1 = M_\mu(z)C_\Pi(z) + \alpha_0 z M_\mu(z). \quad (4.9)$$

where $C_\Pi(z)$ is the function for \mathbb{X} . Now we substitute (4.9) to the equation (4.8) and after a simple computation, we obtain

$$\beta_0 z^2 M_\rho(z) = C_\Pi(z), \quad (4.10)$$

which is equivalent to (4.6) and this completes the proof. \square

Corollary 4.3. *An important benefit of the above theorem is the following one if we know $\varphi(\mathbb{X})$, $\varphi(\mathbb{X}^2)$ and $C_\Pi(z)$ then we also know the measure μ . This result will be applied in the proof of the main theorem.*

Corollary 4.4. *If $\beta_0 = 1$ then $\varphi_\Pi(\mathbb{X}^{n+2})$ is the moment of the variable described by Jacobi parameters (4.5).*

4.3 Some consequences for a two-state normal distribution

Proposition 4.5. *Suppose that we have a self-adjoint variable \mathbb{X} with mean $\varphi(\mathbb{X}) = a$ and second moment $\varphi(\mathbb{X}^2) = b + a^2$. Then the following statements are equivalent:*

1. *the Jacobi parameter is equal to*

$$J(\varphi) = J(\mu) = \begin{pmatrix} a, & 0, & 0, & 0, & \dots \\ b, & 1, & 1, & 1, & \dots \end{pmatrix}, \quad (4.11)$$

2. *the following equation*

$$\varphi_\Pi(\mathbb{X}^{n+2}) = b \int x^n \rho(dx), \quad (4.12)$$

is satisfied for all $n \geq 0$ and ρ is the Wigners semicircle law with mean 0 and variance 1,

3. *$C_\Pi(z)$ satisfies the equation*

$$(C_\Pi(z))^2 - bC_\Pi(z) + b^2 z^2 = 0, \quad (4.13)$$

4. *the relation*

$$M_\mu(z)(b^2 z^2 - b(1 - za) + (1 - za)^2) - 1 + za + b = C_\Pi(z), \quad (4.14)$$

is true.

Proof. $1 \Rightarrow 2$. By Theorem 4.2 we have that $\varphi_{\mathfrak{u}}(\mathbb{X}^{n+2})/b$ is the moment of measure described by Jacobi parameters

$$\begin{pmatrix} 0, & 0, & 0, & 0, & \dots \\ 1, & 1, & 1, & 1, & \dots \end{pmatrix}, \quad (4.15)$$

which gives that this is Wigners semicircle law.

$2 \Rightarrow 3$. The moment generating function of measure ρ given by the Jacobi parameters (4.15) satisfies (see [17]) the equation

$$M_{\rho}^2(z)z^2 - M_{\rho}(z) + 1 = 0. \quad (4.16)$$

The equation (4.12) is equivalent to $C_{\mathfrak{u}}(z) = bz^2M_{\rho}(z)$ and substituting this to the equation (4.16) we get (4.13).

$3 \Rightarrow 4$. If we now apply Lemma 3.11 i.e. $M_{\mu}(z)C_{\mathfrak{u}}(z) = M_{\mu}(z) - 1 - zaM_{\mu}(z)$ to the equation (4.13) we get

$$(M_{\mu}(z) - 1 - zaM_{\mu}(z))C_{\mathfrak{u}}(z) - b(M_{\mu}(z) - 1 - zaM_{\mu}(z)) + b^2z^2M_{\mu}(z) = 0, \quad (4.17)$$

and then we use $M_{\mu}(z) - 1 - zaM_{\mu}(z) = C_{\mathfrak{u}}(z)M_{\mu}(z)$ again and after simple computation we get (4.14).

$4 \Rightarrow 1$. As we have seen in the explanation above, each of the steps above are equivalent so from (4.17) we get (4.12) and taking into account that $\varphi(\mathbb{X}) = a$, $\varphi(\mathbb{Y}) = b + a^2$ and Corollary 4.3 we get that Jacobi parameters for μ are equal to (4.11). \square

4.4 Regression characterization for two-state algebras

The original motivation for this paper is a desire to understand the Theorem 2.1 in the work of Bożejko and Bryc [13]. They proved, that if we have a two-state variable with the same distribution and we assume a two-state analog of conditional quadratic variances (equation (4.18), below), then the corresponding moment generating function satisfies some relationship i.e. (4.19). An open problem in this area is the converse implication to their theorem. In the theorem below we present a stronger version of Theorem 2.1 from the work [13] and the converse implication.

Theorem 4.6. *Suppose \mathbb{X}, \mathbb{Y} are self-adjoint, c -free, $\varphi(\mathbb{X} + \mathbb{Y}) = a$, $\varphi((\mathbb{X} + \mathbb{Y})^2) = b + a^2$, $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ for some $\alpha, \beta > 0$, $\alpha + \beta = 1$ and all integers $k \geq 1$. Let $\mathbb{S} = \mathbb{X} + \mathbb{Y}$, then the following statements are equivalent:*

1. *state φ is connected with “conditional quadratic variances” by the relation*

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2\mathbb{S}^n) = \frac{\alpha\beta}{\tilde{b} + 1}\varphi((\tilde{b}\mathbb{S}^2 + (\tilde{a} - a\tilde{b})\mathbb{S} + \mathbb{I}(b - a\tilde{a}))\mathbb{S}^n), \quad (4.18)$$

2. *the relation between the moment generating function $M_{\mu}(z)$ and $M_{\nu}(z)$ is given by*

$$M_{\mu}(z) = \frac{(\tilde{b} + z\tilde{a})M_{\nu}(z) - \tilde{b} - 1}{M_{\nu}(z)[(b - a\tilde{a})z^2 + z(\tilde{a} - a\tilde{b}) + \tilde{b}] - (\tilde{b} + 1)(1 - az)}, \quad (4.19)$$

3. *the following relation between $C_{\mathfrak{u}}(z)$ and $M_{\nu}(z)$*

$$C_{\mathfrak{u}}(z) = \frac{M_{\nu}(z)z^2b}{-M_{\nu}(z)(\tilde{b} + z\tilde{a}) + \tilde{b} + 1}, \quad (4.20)$$

where $a, b, \tilde{a}, \tilde{b} \in \mathbb{R}$, $\tilde{b} > -1$ and $b > 0$ is satisfied.

Proof. $1 \Rightarrow 2$: Suppose, that the equality (4.18) holds. Thus from (4.18) and Lemma 3.14 we get

$$\alpha\beta\varphi_2(\mathbb{S}^{n+2}) = \frac{\alpha\beta}{\tilde{b}+1}\varphi((\tilde{b}\mathbb{S}^2 + (\tilde{a} - a\tilde{b})\mathbb{S} + \mathbb{I}(b - a\tilde{a}))\mathbb{S}^n). \quad (4.21)$$

A routine argument relates now the power series

$$\begin{aligned} (\tilde{b}+1)C_{\varphi,\psi}^{(2)}(z) &= \tilde{b}M_\mu(z) - \tilde{b}az - \tilde{b} + z((\tilde{a} - a\tilde{b})M_\mu(z) - (\tilde{a} - a\tilde{b})) + z^2(b - a\tilde{a})M_\mu(z) \\ &= M_\mu(z)((b - a\tilde{a})z^2 + z(\tilde{a} - a\tilde{b}) + \tilde{b}) - \tilde{b} - z\tilde{a}. \end{aligned} \quad (4.22)$$

If in (4.22) we multiply both sides by $M_\nu(z)$ and use the fact (3.7) with $R_1(\mathbb{X} + \mathbb{Y}) = a$, we get

$$(\tilde{b}+1)(M_\mu(z) - 1 - azM_\mu(z)) = M_\nu(z)(M_\mu(z)((b - a\tilde{a})z^2 + z(\tilde{a} - a\tilde{b}) + \tilde{b}) - \tilde{b} - z\tilde{a}), \quad (4.23)$$

or equivalently

$$M_\mu(z) = \frac{(\tilde{b} + z\tilde{a})M_\nu(z) - \tilde{b} - 1}{M_\nu(z)[(b - a\tilde{a})z^2 + z(\tilde{a} - a\tilde{b}) + \tilde{b}] - (\tilde{b} + 1)(1 - az)}. \quad (4.24)$$

$2 \Rightarrow 3$: If we use the formula (3.13) with $R_1(X + Y) = a$ to the equation (4.23) we obtain

$$(\tilde{b}+1)C_{\text{II}}(z) = M_\nu(z)((b - a\tilde{a})z^2 + z(\tilde{a} - a\tilde{b}) + \tilde{b}) - (\tilde{b} + z\tilde{a})(1 - C_{\text{II}}(z) - za)M_\nu(z), \quad (4.25)$$

or equivalently

$$C_{\text{II}}(z) = \frac{M_\nu(z)z^2b}{-M_\nu(z)(\tilde{b} + z\tilde{a}) + \tilde{b} + 1}. \quad (4.26)$$

$3 \Rightarrow 1$: Suppose now, that equality (4.20) holds. Applying (3.7) to (4.25) we obtain (4.23). Dividing (4.23) by $M_\nu(z)$ and applying (3.7) we obtain (4.22), which is equivalent to (4.18). \square

Proposition 4.7. *Suppose \mathbb{X}, \mathbb{Y} are self-adjoint, c -free, $\varphi(\mathbb{X} + \mathbb{Y}) = a$, $\varphi((\mathbb{X} + \mathbb{Y})^2) = b + a^2$, $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ for some $\alpha, \beta > 0$, $\alpha + \beta = 1$ and all integers $k \geq 1$. Then the following three statements are equivalent:*

$$(1) \varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2\mathbb{S}^n) = \alpha\beta b\varphi(\mathbb{S}^n), \quad (4.27)$$

$$(2) \varphi((\beta\mathbb{X} - \alpha\mathbb{Y})\mathbb{S}^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta b\psi(\mathbb{S}^n), \quad (4.28)$$

$$(3) \varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})\mathbb{S}^n) = \alpha\beta a\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})^2\mathbb{S}^n), \quad (4.29)$$

for all non-negative integers $n \geq 0$, where $\mathbb{S} = \mathbb{X} + \mathbb{Y}$.

Proof. ($1 \Rightarrow 2$, $1 \Leftarrow 2$): If we put $\tilde{a} = 0$ and $\tilde{b} = 0$ in the equation (4.18), we obtain the relation (4.27) which by Theorem 4.6 is equivalent to

$$C_{\text{II}}(z) = bM_\nu(z)z^2. \quad (4.30)$$

From (4.30) we see that $\varphi_{\text{II}}(\mathbb{S}^{n+2}) = b\psi(\mathbb{S}^n)$. Using Lemma 3.14 we get

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})\mathbb{S}^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta b\psi(\mathbb{S}^n).$$

(1 \Rightarrow 3, 1 \Leftarrow 3): We consider the expression $\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n)$. Since we have that either the first element and the third one are in different blocks, or they are in the same block. In the first case the sum vanishes by (3.15). On the other hand we observe two situations. Firstly, if the first three elements are in the same block then we have $\varphi_3((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n)$. In the second case, if the first element and third one are in the same block but not along with the second one then we get $\varphi_2((\beta\mathbb{X} - \alpha\mathbb{Y})^2(\mathbb{X} + \mathbb{Y})^n)$, but taking into account that $\varphi(\mathbb{X} + \mathbb{Y}) = a$, we get $a\varphi_2((\beta\mathbb{X} - \alpha\mathbb{Y})^2(\mathbb{X} + \mathbb{Y})^n)$. By $\beta R_k(\mathbb{X}) = \alpha R_k(\mathbb{Y})$ and c -free we also obtain (see the proof of Lemmas 3.13 and 3.14)

$$\begin{aligned} & \varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n) \\ &= \varphi_3((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})(\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n) + a\varphi_2((\beta\mathbb{X} - \alpha\mathbb{Y})^2(\mathbb{X} + \mathbb{Y})^n) \\ &= \alpha\beta\varphi_3((\mathbb{X} + \mathbb{Y})^{n+3}) + \alpha\beta a\varphi_2((\mathbb{X} + \mathbb{Y})^{n+2}). \end{aligned} \quad (4.31)$$

The equation (4.27) is equivalent to $C_{\varphi,\psi}^{(2)}(z) = bz^2M_\mu(z)$ where $C_{\varphi,\psi}^{(2)}(z)$ is a function for $\mathbb{X} + \mathbb{Y}$ (see Theorem 4.6). So from (3.8) we have $M_\nu(z)C_{\varphi,\psi}^{(3)}(z) = 0$ (because $R_2(\mathbb{X} + \mathbb{Y}) = b$). But $M_\nu(z) \neq 0$ for sufficiently small $|z|$, which gives $C_{\varphi,\psi}^{(3)}(z) = 0$ thus we obtain (4.29) because $\varphi_3((\mathbb{X} + \mathbb{Y})^{n+3}) = 0$. If now (4.29) holds then $C_{\varphi,\psi}^{(3)}(z) = 0$ and by (3.8) we easily get (4.27). \square

5 Proof of the main theorem

From Proposition 4.7 we see that it is sufficient to show that 1 \Rightarrow 2 and 1 \Leftarrow 2.

Proof. 1 \Rightarrow 2: Suppose that \mathbb{X} and \mathbb{Y} have the two-state normal laws. Let's denote $\mu = \mu_1 \boxplus \mu_2$ and $\nu = \nu_1 \boxplus \nu_2$ then from Lemma 3.16 we get that the measure μ and ν have Jacobi parameters (2.16) and (2.17), respectively. Using Proposition 4.5 we obtain equation (4.14) with $C_{\mathfrak{u}}(z)$ for $\mathbb{X} + \mathbb{Y}$. Expanding $M(z)$ and $C_{\mathfrak{u}}(z)$ in the series and using the fact from the equation (3.22) (we skip a simple computation) we get

$$\varphi((\beta\mathbb{X} - \alpha\mathbb{Y})(\mathbb{X} + \mathbb{Y})^n(\beta\mathbb{X} - \alpha\mathbb{Y})) = \alpha\beta\varphi((1-b)\mathbb{S}^2 + (-2a + ab)\mathbb{S} + (a^2 + b^2)\mathbb{I}).$$

Now we prove (2.18). The moment generating function for $\mathbb{X} + \mathbb{Y}$ with respect to φ satisfies (by the equation (2.14))

$$zM_\mu(z) = \frac{1}{\frac{1}{z} - a - \frac{b}{\frac{1}{z} - zM_\sigma(z)}}, \quad (5.1)$$

where $M_\sigma(z)$ is the moment generating function for the measure σ with Jacobi parameters

$$\begin{pmatrix} 0, & 0, & 0, & \dots \\ 1, & 1, & 1, & \dots \end{pmatrix}. \quad (5.2)$$

The equation (5.1) is equivalent to

$$(M_\mu(z)(1 - za) - 1) - (M_\mu(z)(1 - za) - 1)z^2M_\sigma(z) = z^2bM_\mu(z), \quad (5.3)$$

From Lemma 3.5 we have $M_\mu(z) - 1 - zaM_\mu(z) = M_\nu(z)C_{\varphi,\psi}^{(2)}(z)$, so

$$(M_\mu(z)(1 - za) - 1) - M_\nu(z)C_{\varphi,\psi}^{(2)}(z)z^2M_\sigma(z) = z^2bM_\mu(z), \quad (5.4)$$

but $M_\nu(z)M_\sigma(z)z^2 = M_\nu(z) - 1$ (see the equation (4.8)), so

$$(M_\mu(z) - azM_\mu(z) - 1) - (M_\nu(z) - 1)C_{\varphi,\psi}^{(2)}(z) = z^2bM_\mu(z), \quad (5.5)$$

then we use again $M_\mu(z) - 1 - zaM_\mu(z) = M_\nu(z)C_{\varphi,\psi}^{(2)}(z)$ and we get

$$C_{\varphi,\psi}^{(2)}(z) = z^2bM_\mu(z), \quad (5.6)$$

which is equivalent to (2.18), because $\alpha\beta\varphi_2((\mathbb{X} + \mathbb{Y})^{n+2}) = \varphi((\alpha\mathbb{Y} - \beta\mathbb{X})^2(\mathbb{X} + \mathbb{Y})^n)$.

$2 \Rightarrow 1$: Suppose now that the equalities (2.18) and (2.19) hold. The relation (2.19) is equivalent to (4.14). From Proposition 4.5 we deduce that the Jacobi parameters for μ (i.e. $\mathbb{X} + \mathbb{Y}$) is given by (4.11), and $C_{\parallel}(z)$ satisfies the equation

$$(C_{\parallel}(z))^2 - bC_{\parallel}(z) + b^2z^2 = 0.$$

From Theorem 4.6 we obtain that (2.18) is equivalent to $C_{\parallel}(z) = M_\nu(z)bz^2$ so we get

$$z^2M_\nu(z) - M_\nu(z) + 1 = 0.$$

The equation above is equivalent to

$$M_\nu(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2}. \quad (5.7)$$

It is well known that the measure ν have Jacobi parameters (2.17) (see [12, 33]). From the assumption on cumulants and c -free we see $r_k(\mathbb{X}) = r_k(\mathbb{X} + \mathbb{Y})/\beta$ and $r_k(\mathbb{Y}) = r_k(\mathbb{X} + \mathbb{Y})/\alpha$ i.e. cumulants disappear for $k > 2$ (of course we deduce similarly for $R_k(\mathbb{X})$ and $R_k(\mathbb{Y})$). Thus we see that \mathbb{X} and \mathbb{Y} have two-state normal distributions which proves the theorem. \square

Open problems and remarks

- In this paper we assume that the measures μ and ν have compact supports. It would be interesting to show if this measures can be replaced by any probability measure.
- A version of Theorem 2.8 can be extended for two-state Meixner random variable (see [7]). The proof of this theorem is analogous to the proof of Theorem 2.8 (technical and demanding tools in this article).
- It would be interesting to show that Theorem 2.8 above is true for two-state free Brownian motion. The existence of such process, far from being trivial, is ensured by Anshelevich [6].

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